

ON  $p$ -EMBEDDING PROBLEMS IN CHARACTERISTIC  $p$ 

LIOR BARY-SOROKER AND NGUYỄN DUY TÂN

ABSTRACT. Let  $K$  be a valued field of characteristic  $p > 0$  with non- $p$ -divisible value group. We show that every finite embedding problem for  $K$  whose kernel is a  $p$ -group is properly solvable.

AMS Mathematics Subject Classification (2010): 12E30, 12F12

## 1. INTRODUCTION

In the proof that every finite solvable group occurs as a Galois group over the rationals, Shafarevich studies the solvability of embedding problems with nilpotent kernel and solvable cokernel. To study the absolute Galois group  $\text{Gal}(K)$  of a field  $K$  via embedding problems became a trend in recent papers, e.g. [BHH, Ha3, HS, Pa1, Po]. See also the upcoming book [Ja] and references therein.

In this work we consider a field  $K$  of characteristic  $p > 0$  and the finite embedding problems for  $K$  whose kernels are  $p$ -groups which we call **finite  $p$ -embedding problems**. An obvious necessary condition to have a *proper* solution is to have a *weak* solution (see Section 3 for definitions). This latter condition is automatically satisfied in our case, since  $\text{cd}_p(\text{Gal}(K)) \leq 1$ , for a field of characteristic  $p > 0$ . We obtain a mild sufficient condition on  $K$  to have a proper solution of any finite  $p$ -embedding problem.

**Theorem 1.1.** *Let  $K$  be a field of characteristic  $p$  admitting a non- $p$ -divisible valuation. Then every finite  $p$ -embedding problem for  $K$  is solvable.*

Some examples of fields satisfying this condition are the following. If  $R$  is a Noetherian domain or Krull domain of characteristic  $p > 0$ , then its fraction field  $K$  satisfies the hypothesis of Theorem 1.1. If  $R$  is an arbitrary domain of characteristic  $p > 0$ , then the fraction fields of the ring  $R[x_1, \dots, x_n]$  of polynomials and of the ring of formal Taylor series  $R[[x_1, \dots, x_n]]$  satisfy the hypothesis of Theorem 1.1, for any  $n \geq 1$ .

The proof of Theorem 1.1 is based on the following cohomological criterion of Harbater. A profinite group  $\Pi$  is called **strongly  $p$ -dominating** if  $H^1(\Pi, P)$  is infinite for every nontrivial finite elementary  $p$ -group  $P$  on which  $\Pi$  acts<sup>1</sup>.

**Theorem 1.2** ([Ha2, Theorem 1b]). *Let  $\Pi$  be a profinite group. Assume that  $\Pi$  is strongly  $p$ -dominating and that  $\text{cd}_p(\Pi) \leq 1$ . Then every finite  $p$ -embedding problem for  $\Pi$  is properly solvable.*

---

*Date:* August 11, 2010.

The second author is partially supported by NAFOSTED, SFB/TR45 and the ERC/Advanced Grant 226257.

<sup>1</sup>All actions, homomorphism, etc., in this work are assumed to be continuous.

Harbater's motivation for Theorem 1.2 is to show that every finite  $p$ -embedding problem for the étale fundamental group  $\Pi := \pi_1(X)$  of an affine variety  $X$  over an arbitrary field  $K$  of characteristic  $p > 0$  has a proper solution [Ha1].

We show that the converse of Theorem 1.2 also holds true, see Theorem 4.2. Moreover, to get the assertion of Theorem 1.2, one may suspect that the infinitude of  $H^1(\Pi, \mathbb{Z}/p\mathbb{Z})$  suffices, where  $\Pi$  acts trivially on  $\mathbb{Z}/p\mathbb{Z}$ . This is true if both the kernel and cokernel are  $p$ -groups, but in general it fails, see [Ha2].

By Theorem 1.2, to prove Theorem 1.1 it suffices to show that  $\text{Gal}(K)$  is strongly  $p$ -dominating. This is carried out by using that for every nontrivial finite elementary  $p$ -group  $P$  on which  $\Pi$  acts we have  $H^1(K, P) = K/f(K)$ , for some additive polynomial  $f$  (Lemma 3.1). Then using the non- $p$ -divisible valuation of  $K$  we construct infinitely many  $a \in K$  that are distinct modulo  $f$ .

We conclude the introduction with an example. Let  $K_0$  be a field of characteristic  $p > 0$  and  $K = K_0((x))$  the field of formal Laurent series. Then by Theorem 1.1 every finite  $p$ -embedding problem is properly solvable. When  $K_0$  is algebraically closed, Harbater proves this in [Ha2, Example 5] using a similar method. However, when  $K_0$  is arbitrary Harbater invokes a theorem of Katz-Gabber in order to complete his proof (see [Ha2, Proposition 6]).

**Acknowledgements:** We would like to give our sincere thanks to Hélène Esnault for her support and constant encouragement.

The first author is an Alexander von Humboldt fellow in The Institut für Experimentelle Mathematik, Universität Duisburg-Essen.

## 2. VALUATION-THEORETIC LEMMAS

Let  $A$  be a ring. By a **valuation** of  $A$  we shall mean a map  $v : x \mapsto v(x)$  of  $A$  onto a totally ordered commutative group  $\Gamma$  (written additively), together with an extra element  $\infty$ , such that:

- (1)  $\alpha + \infty = \infty$  and  $\alpha < \infty$  for all  $\alpha \in \Gamma$ .
- (2)  $v(x) = \infty$  if and only if  $x = 0$ .
- (3)  $v(xy) = v(x) + v(y)$  for all  $x, y \in A$ .
- (4)  $v(x + y) \geq \min\{v(x), v(y)\}$ .

If  $A$  is a ring with a valuation  $v$  on  $A$ , we shall also say simply that  $A$  is a **valued ring**. The group  $\Gamma$  is called the **value group**.

**Lemma 2.1.** *Let  $\Gamma$  be a nontrivial totally ordered commutative group*

- (1) *For any element  $\gamma$  in  $\Gamma$ , there exists  $\beta \in \Gamma$  such that  $\beta < \gamma$ .*
- (2) *Let  $\gamma_1, \dots, \gamma_r$  be elements in  $\Gamma$  and let  $n_1, \dots, n_r$  be positive numbers. Then there exists an element  $\gamma_0$  in  $\Gamma$  such that for all elements  $\gamma < \gamma_0$ ,  $\gamma \in \Gamma$ , we have  $n_i \gamma < \gamma_i$  for all  $i$ .*

*Proof.* 1) If  $\gamma \geq 0$ , then let  $\beta < 0 \leq \gamma$  (such an element exists since  $\Gamma$  is nontrivial).

If  $\gamma < 0$ , one can take  $\beta = 2\gamma < \gamma$ .

2) We set

$$\gamma_0 := \min\{\gamma_1, \dots, \gamma_r, 0\}.$$

Now let  $\gamma$  be an arbitrary element such that  $\gamma < \gamma_0$ . Since  $\gamma < \gamma_i$ ,  $\gamma < 0$ , it follows that  $n_i\gamma < \gamma_i$ , for all  $i$ .  $\square$

**Lemma 2.2.** *Let  $A$  be a valued ring of characteristic  $p > 0$  with nontrivial value group  $\Gamma$ . Let  $f(T) = b_0T + \dots + b_mT^{p^m}$  be a  $p$ -polynomial in one variable with coefficients in  $A$ . Then there exists an element  $\gamma_0 \in \Gamma$  such that if  $a = f(a_1)$ ,  $a_1 \in A$  and  $v(a) < \gamma_0$  then  $v(a) = v(b_m) + p^mv(a_1)$ .*

*Proof.* By Lemma 2.1, there exists an element  $\alpha \in \Gamma$  such that for all  $\gamma < \alpha$  in  $\Gamma$ , we have

$$(p^m - p^i)\gamma < v(b_i) - v(b_m), \quad \forall 0 \leq i < m.$$

We set

$$\beta := \min\{v(b_i) + \alpha p^i \mid 0 \leq i \leq m\}.$$

Let  $\gamma_0$  be any element with  $\gamma_0 < \beta$ . Now assume that  $a = f(a_1)$  such that  $v(a) < \gamma_0$  ( $a_1 \in A$ ). Let  $s$  be an index such that

$$v(b_s a_1^{p^s}) = \min\{v(b_i a_1^{p^i}) \mid 0 \leq i \leq m\}.$$

Then

$$v(b_s) + p^s\alpha > \gamma_0 > v(f(a_1)) \geq v(b_s) + p^sv(a_1).$$

Thus  $0 < p^s(\alpha - v(a_1))$  and hence  $v(a_1) < \alpha$ . By the choice of  $\alpha$ , we have

$$v(b_i a_1^{p^i}) = v(b_i) + p^iv(a_1) > v(b_m) + p^mv(a_1) = v(b_m a_1^{p^m}), \quad \forall i < m.$$

Therefore  $v(a) = v(b_m) + p^mv(a_1)$  as required.  $\square$

**Lemma 2.3.** *Let  $\Gamma$  be a non- $p$ -divisible totally ordered commutative group. Let  $\alpha_0, \gamma_0$  be elements in  $\Gamma$ . Then there exist infinitely many elements  $\gamma_i \in \Gamma$  such that*

$$\gamma_0 > \gamma_1 > \dots > \gamma_i > \dots$$

*and  $\gamma_i \notin \alpha_0 + p\Gamma$ , for all  $i > 0$ .*

*Proof.* We first consider the case  $\alpha_0 = 0$ . Since  $\Gamma$  is not  $p$ -divisible, there is an element  $a_0 \in \Gamma$  such that  $a_0 \notin p\Gamma$ . By Lemma 2.1 part (2), there exists an element  $\delta_0$  such that for all  $\delta < \delta_0$ , we have  $p\delta < \gamma_0 = a_0$ . By Lemma 2.1 part (1), there exists an infinite sequence

$$\delta_0 > \delta_1 > \dots > \delta_i > \dots$$

Set  $\gamma_i := a_0 + p\delta_i$ , for all  $i > 0$ . Then  $\gamma_i \notin p\Gamma$ , for all  $i > 0$  and  $\gamma_0 > \gamma_1 > \dots > \gamma_i > \dots$ .

For the general case, applying the previous argument for  $\gamma'_0 := \gamma_0 - \alpha_0$ , we get an infinite sequence  $\gamma'_0 > \gamma'_1 > \dots > \gamma'_i > \dots$  with  $\gamma'_i \in \Gamma$  but  $\gamma'_i \notin p\Gamma$ . Setting  $\gamma_i := \gamma'_i + \alpha_0$ , we get a desired sequence of elements.  $\square$

**Proposition 2.4.** *Let  $A$  be a valued ring of characteristic  $p > 0$  with non- $p$ -divisible value group  $\Gamma$ . Let  $f(T) = b_0T + b_1T^p + \cdots + b_mT^{p^m}$  be a  $p$ -polynomial in one variable with coefficients in  $A$  with  $m \geq 1$  and  $b_m \neq 0$ . Then  $A/f(A)$  is infinite.*

*Proof.* Let  $\gamma_0$  be as in Lemma 2.2. For any  $a$  in  $A$  such that  $v(a) < \gamma_0$  and  $v(a) \notin v(b_m) + p\Gamma$ , Lemma 2.2 implies that  $a$  is not in  $f(A)$ . By Lemma 2.3 and noting that the valuation map  $v$  is onto, we may choose a sequence  $\{a_i\}$  of elements from  $A$  such that  $v(a_i) \notin v(b_m) + p\Gamma$  for all  $i$ , and  $\gamma_0 > v(a_1) > v(a_2) > \cdots > v(a_i) > \cdots$ . For every  $i < j$ , one has

$$v(a_i - a_j) = v(a_i) \notin v(b_m) + p\Gamma,$$

so  $a_i - a_j \notin f(A)$  and hence  $a_i, a_j$  have different images in  $A/f(A)$ . Therefore,  $A/f(A)$  is infinite.  $\square$

### 3. PROOF OF THEOREM 1.1 AND A COROLLARY

An **embedding problem**  $\mathcal{E}$  for a profinite group  $\Pi$  is a diagram

$$\begin{array}{ccc} & \Pi & \\ & \downarrow \alpha & \\ \Gamma & \xrightarrow{f} & G \end{array}$$

which consists of a pair of profinite groups  $\Gamma$  and  $G$  and epimorphisms  $\alpha : \Pi \rightarrow G$ ,  $f : \Gamma \rightarrow G$ .

A **weak solution** of  $\mathcal{E}$  is a homomorphism  $\beta : \Pi \rightarrow \Gamma$  such that  $f\beta = \alpha$ . If such a  $\beta$  is surjective, then it is called a **proper solution**. We will call  $\mathcal{E}$  **weakly** (resp. **properly**) **solvable** if it has a weak (resp. proper) solution.

We call  $\mathcal{E}$  a **finite** embedding problem if the group  $\Gamma$  is finite.

The **kernel** of  $\mathcal{E}$  is defined to be  $N := \ker(f)$ . We call  $\mathcal{E}$  a **p-embedding problem** if  $N$  is a  $p$ -group.

We say  $\mathcal{E}$  is a **split** embedding problem if  $f : \Gamma \rightarrow G$  has a group theoretical section, i.e.,  $f' : G \rightarrow \Gamma$  such that  $ff'$  is the identity map on  $G$ .

In this note, by a  **$K$ -group**, where  $K$  is a field, we mean an algebraic affine group scheme which is smooth ([Wa]). This notion is equivalent to the notion of a linear algebraic group defined over  $K$  in the sense of [Bo].

First we need the following lemma.

**Lemma 3.1.** *Let  $K$  be an infinite field of characteristic  $p > 0$ . Let  $P$  be a nontrivial finite commutative  $K$ -group which is annihilated by  $p$ . Then  $G$  is  $K$ -isomorphic to a  $K$ -subgroup of the additive group  $\mathbb{G}_a$ , of the form  $\{x \mid f(x) = 0\}$ , where  $f(T) = T + b_1T^p + \cdots + b_mT^{p^m}$  is a  $p$ -polynomial with coefficients in  $K$ ,  $m \geq 1$  and  $b_m \neq 0$ .*

*Proof.* This is well known, see e.g. [CGP, Proposition B.1.13] or [Oe, Chapitre V, Proposition 4.1 and Subsection 6.1].  $\square$

We are now ready to prove Theorem 1.1.

*Proof of Theorem 1.1.* We have  $\text{cd}_p(\text{Gal}(K)) \leq 1$  (see, e.g., [Se2, Chapter II, Proposition 3]). By Theorem 1.2 it suffices to prove that  $\text{Gal}(K)$  is strongly  $p$ -dominating.

Indeed, let  $P$  be a non-trivial elementary  $p$ -group on which  $\text{Gal}(K)$  acts. Consider  $P$  as a finite  $K$ -group. Then  $P$  is commutative and annihilated by  $p$ . Hence by Lemma 3.1,  $P$  is  $K$ -isomorphic to a subgroup of  $\mathbb{G}_a$  defined as the kernel of  $f : \mathbb{G}_a \rightarrow \mathbb{G}_a$ , where  $f(T) = T + \cdots + b_m T^{p^m}$  is a  $p$ -polynomial in one variable with coefficients in  $K$  with  $m \geq 1$  and  $b_m \neq 0$ . We have the following exact sequence of  $K$ -groups

$$0 \rightarrow P \rightarrow \mathbb{G}_a \xrightarrow{f} \mathbb{G}_a \rightarrow 0.$$

From this exact sequence we get the following exact sequence of Galois cohomology groups

$$H^0(K, \mathbb{G}_a) \xrightarrow{f} H^0(K, \mathbb{G}_a) \rightarrow H^1(K, P) \rightarrow H^1(K, \mathbb{G}_a).$$

By Hilbert 90  $H^1(K, \mathbb{G}_a) = 0$  (see e.g. [Se2, Chapter II, Proposition 1]), hence

$$H^1(K, P) \simeq H^0(K, \mathbb{G}_a)/\text{im}(f) = K/f(K).$$

The latter is infinite by Proposition 2.4. So we conclude that  $H^1(K, P)$  is infinite, and hence  $\text{Gal}(K)$  is strongly  $p$ -dominating.  $\square$

We recall that a Hilbertian field is a field  $K$  having the irreducible specialization property: for every irreducible polynomial  $f(T, X) \in k[T, X]$  that is separable in  $X$ , there exists  $a \in K$  such that  $f(a, X)$  is irreducible in  $k[X]$  (we refer readers to [FJ, Chapters 12, 13] for more details about Hilbertian fields). In [DD], Dèbes and Deschamps give the following conjecture.

**Conjecture 3.2** ([DD, 2.1.2]). *Let  $K$  be a Hilbertian field. Then every finite split embedding problem for  $\text{Gal}(K)$  has a proper solution.*

An easy consequence of Theorem 1.1 is a simple proof of [MM, Theorem 8.3] which asserts that Conjecture 3.2 holds true whenever  $K$  is of characteristic  $p > 0$  and if the kernel of the embedding problem is a  $p$ -group. Namely, we have

**Corollary 3.3.** *Let  $K$  be a Hilbertian field of characteristic  $p > 0$ . Then every finite  $p$ -embedding problem for  $\text{Gal}(K)$  is properly solvable.*

*Proof.* Let  $\mathcal{E} = (\alpha : \text{Gal}(K) \rightarrow A, f : B \rightarrow A)$  be a finite  $p$ -embedding problem for  $\text{Gal}(K)$ . Consider the finite  $p$ -embedding problem  $\mathcal{E}_t := (\alpha \circ \text{pr}_t : \text{Gal}(K(t)) \rightarrow A, f : B \rightarrow A)$  for  $\text{Gal}(K(t))$  obtained by composition with the restriction map  $\text{Gal}(K(t)) \rightarrow \text{Gal}(K)$ . Since  $K(t)$  has discrete valuations, Theorem 1.1 gives a proper solution of  $\mathcal{E}_t$ , say  $\theta_t : \text{Gal}(K(t)) \rightarrow B$ . By the irreducible specialization property (applied to a polynomial a root of which generates the solution field of  $\theta_t$  over  $K(t)$ )  $\theta_t$  specializes to a proper solution  $\theta$  of  $\mathcal{E}$  (see [FJ, Lemma 16.4.2]).  $\square$

**Remark 3.4.** (1) Let  $G$  be a finite  $p$ -group,  $K$  a Hilbertian field of characteristic  $p > 0$ . By considering the finite (split)  $p$ -embedding problem  $(\text{Gal}(K) \rightarrow \{1\}, G \rightarrow \{1\})$ , Corollary 3.3 implies that  $G$  is realizable over  $K$ . In other words, this proposition shows that every finite  $p$ -group is realizable over an arbitrary Hilbertian field of

characteristic  $p > 0$ . This last statement is a special case of a theorem of Shafarevich, [FJ, Theorem 16.4.7].

- (2) Corollary 3.3 can also be derived from Ikeda's theorem ([FJ, Proposition 16.4.5]). Here we sketch the proof: one starts with a finite embedding problem for  $K$  corresponding to an exact sequence  $1 \rightarrow P \rightarrow B \rightarrow A \rightarrow 1$ , where  $P$  is a  $p$ -group and  $B = \text{Gal}(L/K)$ . We use the usual trick of decomposing this embedding problem to a series of embedding problems in order to assume that  $P$  is a minimal normal subgroup of  $B$ . In particular  $P$  is *abelian*. Since  $\text{cd}_p(K) \leq 1$  we can replace this embedding problem by a bigger *split* embedding problem with the same kernel by taking the fiber product of  $B$  and the image of a weak solution. Now we use Ikeda's result that gives a *regular* solution over  $K$ , i.e., a solution over  $K(t)$  with the extra condition that the solution field is regular over  $L$ . Then one uses Hilbertianity to reduce the solution to a solution over  $K$ .

Unfortunately, we do not know whether any finite  $p$ -embedding problem over a field of characteristic  $p > 0$  has a regular solution.

- (3) For recent results concerning Conjecture 3.2, we refer readers to [BP, Pa1, Pa2, Po].

#### 4. EMBEDDING PROBLEMS WITH $p$ -KERNEL

In this section we show that the converse of Theorem 1.1 also holds true, see Theorem 4.2. Let

$$\mathcal{E} := \begin{array}{ccccccc} & & & & \Pi & & \\ & & & & \downarrow \alpha & & \\ 1 & \longrightarrow & P & \longrightarrow & \Gamma & \xrightarrow{f} & G \longrightarrow 1 \end{array}$$

be an embedding problem for  $\Pi$  with abelian kernel  $P$ . Since  $P$  is abelian, there is an induced conjugation action of  $G$  on  $P$  by choosing representatives in  $\Gamma$ . This in turn yields an action of  $\Pi$  on  $P$  via  $\alpha : \Pi \rightarrow G$ . Let  $H^1(\Pi, P)$  be the corresponding Galois cohomology group.

Two weak solutions  $\beta$  and  $\beta' : \Pi \rightarrow \Gamma$  of  $\mathcal{E}$  are defined to be equivalent, and denoted by  $\beta \sim \beta'$ , if there is an element  $p$  in  $P$  such that  $\beta' = \text{inn}(p) \circ \beta$ . (Here  $\text{inn}(p) \in \text{Aut}(\Gamma)$  denotes left conjugation by  $p$ .) One can check that  $\sim$  is an equivalence on the set of weak solutions to  $\mathcal{E}$ . Denote by  $\text{WS}(\mathcal{E})$  the set of weak solutions of  $\mathcal{E}$  modulo the equivalence relation  $\sim$ . We have a cohomological description of  $\text{WS}(\mathcal{E})$ .

**Lemma 4.1.** *With notation as above, assume that  $\mathcal{E}$  is weakly solvable. Then  $\text{WS}(\mathcal{E})$  is a  $H^1(\Pi, P)$ -torsor. In particular, any weak solution  $\theta$  of  $\mathcal{E}$  induces a bijection*

$$\text{WS}(\mathcal{E}) \cong H^1(\Pi, P).$$

*Proof.* See [NSW, Proposition 9.4.4]. □

Next we prove the converse of Theorem 1.2. For future reference we formulate it as an if and only if theorem.

**Theorem 4.2.** *Let  $\Pi$  be a profinite group. Then every finite  $p$ -embedding problem for  $\Pi$  has a proper solution if and only if  $\text{cd}_p(\Pi) \leq 1$  and  $\Pi$  is strongly  $p$ -dominating.*

*Proof.* ( $\Leftarrow$ ): This is Theorem 1.2.

( $\Rightarrow$ ): It suffices to prove that  $\Pi$  is strongly  $p$ -dominating. Let  $P$  be a nontrivial elementary abelian  $p$ -group on which  $\Pi$  acts continuously. We have to show that  $H^1(\Pi, P)$  is infinite.

Since the action of  $\Pi$  on  $P$  is continuous, it factors via a finite quotient. I.e., there is a map  $\alpha: \Pi \rightarrow G$  and an action of  $G$  on  $P$  that induces the action of  $\Pi$  on  $P$ . Let  $\Gamma$  be the semidirect product of  $P$  and  $G$ . We get the following split embedding problem with elementary abelian  $p$ -kernel

$$\mathcal{E} := \begin{array}{ccccccc} & & & & \Pi & & \\ & & & & \downarrow \alpha & & \\ 1 & \longrightarrow & P & \longrightarrow & \Gamma & \xrightarrow{f} & G \longrightarrow 1 \end{array}$$

For any  $n > 0$  let  $\Gamma_G^n$  be the  $n$ -th fold fiber product of  $\Gamma$  over  $G$ , i.e.,

$$\Gamma_G^n = \{(\gamma_1, \dots, \gamma_n), \gamma_i \in \Gamma, \text{ and } f(\gamma_1) = \dots = f(\gamma_n) \in G\}.$$

We have a map  $f_n: \Gamma_G^n \rightarrow G$ , defined by  $f_n((\gamma_i)_{i=1}^n) = f(\gamma_1)$ .

We have an embedding problem  $\mathcal{E}_n$  for  $\Pi$  corresponding to the exact sequence

$$1 \longrightarrow P^n \longrightarrow \Gamma_G^n \xrightarrow{f_n} G \longrightarrow 1.$$

By assumption, there is a proper solution  $\beta$  to  $\mathcal{E}_n$ . By composing  $\beta$  with the projections  $\text{pr}_i: \Gamma_G^n \rightarrow \Gamma$ , we get  $n$  proper solutions  $\beta_1, \dots, \beta_n$ .

We show that these  $\beta_i$  are pairwise non-equivalent (and in particular distinct). Indeed, if  $\beta_i \sim \beta_j$ , for some  $1 \leq i < j \leq n$ , then there is a element  $p \in P$  such that  $\beta_i(s) = p\beta_j(s)p^{-1}$ , for all  $s \in \Pi$ . Since  $P$  is a non-trivial group, we can take two different elements  $q, q'$  from  $P$ . Set  $x = (1, \dots, q, \dots, q', \dots, 1) \in \Gamma^n$ , where  $q, q'$  are in  $i$ -th and  $j$ -th entry, respectively and 1 is in all other entries. Then  $x \in \Gamma_G^n$ . Since  $\beta$  is a *proper* solution, there exists  $s$  in  $\Pi$  such that  $\beta(s) = x$ . We then have

$$q = \beta_i(s) = p\beta_j(s)p^{-1} = pq'p^{-1} = q',$$

a contradiction.

Therefore, we get that  $\text{WS}(\mathcal{E})$  is infinite, and by Lemma 4.1,  $H^1(\Pi, P)$  is infinite, as needed.  $\square$

## REFERENCES

- [BHH] L. Bary-Soroker, D. Haran, and D. Harbater, Permanence criteria for semi-free profinite groups, Math. Ann. (to appear).
- [BP] L. Bary-Soroker and E. Paran, Fully Hilbertian fields, preprint.
- [Bo] A. Borel, Linear Algebraic Groups (2nd ed.), Graduate texts in mathematics **126**, New York: Springer-Verlag 1991.
- [CGP] B. Conrad, O. Gabber and G. Prasad, Pseudo-reductive groups, Series: New Mathematical Monographs (No. 17) (to appear).

- [DD] P. Dèbes and B. Deschamps, The regular inverse Galois problem over large fields, in *Geometric Galois Actions*, vol. 2, London Math. Soc. Lecture Note Ser. **243**, Cambridge Univ. Press, Cambridge, 1997, pp. 119–138.
- [FJ] M. D. Fried and M. Jarden, Field arithmetic, third ed., *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*, vol. 11, Springer-Verlag, Berlin, 2008, Revised by Jarden.
- [Ha1] D. Harbater, Embedding problems with local conditions, *Israel Journal of Mathematics*, **118** (2000), 317–355.
- [Ha2] D. Harbater, Correction and addendum to “Embedding problems with local conditions”, *Israel J. Math.* **162** (2007), 373–379.
- [Ha3] D. Harbater, On function fields with free absolute galois groups, *Journal für die reine und angewandte Mathematik (Crelles Journal)*, **2009**(632)(2009) 85–103 .
- [HS] D. Harbater and K. F. Stevenson, Local Galois theory in dimension two, *Advances in Mathematics*, **198**(2)(2005), 623–653.
- [Ja] M. Jarden, Algebraic patching, *Springer Monographs in Mathematics* (to appear).
- [Ka] N. Katz, Local-to-global extensions of representations of fundamental groups, *Ann. Inst. Fourier, Grenoble* **36** (1986), 69–106.
- [MM] G. Malle and B. H. Matzat, Inverse Galois theory, *Springer Monographs in Mathematics*, 1999.
- [NSW] J. Neukirch, A. Schmidt and K. Wingberg, Cohomology of Number Fields, *Grundlehren der mathematischen Wissenschaften Bd. 323*, Springer 2000.
- [Oe] J. Oesterlé, Nombre de Tamagawa et groupes unipotents en caractéristique  $p$ . *Invent. Math.* **78** (1984), 13–88.
- [Pa1] E. Paran, Split embedding problems over complete domains, *Annals of Math.*, **170** (2009), 899–914.
- [Pa2] E. Paran, Galois theory over complete local domains, *Math. Annalen* (to appear).
- [Po] F. Pop, Henselian implies large, *Annals of Math.* (to appear).
- [Se1] J.-P. Serre, Construction de revêtements étales de la droite affine en caractéristique  $p$ , *C.R. Acad. Sci. Paris, Sér. I, Math.*, **311** (1990), 341–346.
- [Se2] J.-P. Serre, Galois cohomology, Corr. 2 printing; Springer 2002 (*Springer Monographs in Mathematics*).
- [Wa] W. C. Waterhouse, Introduction to affine group schemes, *Graduate texts in mathematics* **66**, New York: Springer-Verlag 1979.

INSTITUT FÜR EXPERIMENTELLE MATHEMATIK, UNIVERSITÄT DUISBURG-ESSEN, ELLERNSTRASSE 29, D-45326 ESSEN, GERMANY

*E-mail address:* lior.bary-soroker@uni-due.de

UNIVERSITÄT DUISBURG-ESSEN, FB6, MATHEMATIK, 45117 ESSEN, GERMANY, AND INSTITUTE OF MATHEMATICS, 18 HOANG QUOC VIET, 10307, HANOI - VIETNAM.

*E-mail address:* duy-tan.nguyen@uni-due.de